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Holographic gravitational anomalies

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Abstract: In the AdS/CFT correspondence one encounters theories that are not invariant under diffeomorphisms. In the boundary theory this is a gravitational anomaly, and can arise in $4k+2$ dimensions. In the bulk, there can be gravitational Chern-Simons terms which vary by a total derivative. We work out the holographic stress tensor for such theories, and demonstrate agreement between the bulk and boundary. Anomalies lead to novel effects, such as a nonzero angular momentum for global AdS_3. In string theory such Chern-Simons terms are known with exact coefficients. The resulting anomalies, combined with symmetries, imply corrections to the Bekenstein-Hawking entropy of black holes that agree exactly with the microscopic counting.

Keywords: AdS-CFT Correspondence, Anomalies in Field and String Theories, Black Holes in String Theory, Chern-Simons Theories.

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1. Introduction

Given a theory of gravity in $d+1$ dimensional anti-de Sitter spacetime, it is possible to define a boundary stress tensor as the variation of the on-shell action with respect to the metric on the conformal boundary of the spacetime,

$$\delta S = \frac{1}{2}\int d^d x \sqrt{g^{(0)}}\, T^{ij} \delta g^{(0)}_{ij} \,. \tag{1.1}$$

Here $g^{(0)}_{ij}$ is the metric on the conformal boundary, as reviewed below. The AdS/CFT correspondence asserts that T^{ij} is the expectation value of the stress tensor in a CFT defined on a space with metric conformal to $g^{(0)}_{ij}$. This relation has been the subject of much work, e.g. [1–6].

If the gravitational action is diffeomorphism invariant we will have $\delta S = 0$ for $\delta g^{(0)}_{ij} = \nabla_{(i}\xi_{j)}$. Inserting this into (1.1) and integrating by parts we conclude that the stress tensor is conserved: $\nabla_i T^{ij} = 0$. But this is clearly not the most general situation, since quantum field theories in $4k+2$ dimensions can have gravitational anomalies rendering the stress tensor non-conserved [7–9] (see [10] for a pedagogical review). For example, this is the situation for CFTs in two dimensions with unequal left and right moving central charges. Gravitational anomalies do not spoil the consistency of a quantum field theory, but they do make it impossible to consistently couple the theory to dynamical gravity.

The AdS/CFT correspondence thus forces us to confront non-diffeomorphism invariant theories of gravity in the bulk in order to account for the non-conservation of the boundary stress tensor. Such theories will be inconsistent unless the non-invariance is of a very special type, namely a pure boundary term. Variation of the action by a boundary term is harmless since it does not affect the local dynamics, and is just what we need to agree with the gravitational anomaly of the boundary theory. At the two-derivative order the bulk action is described by the Einstein-Hilbert term supplemented by boundary terms, and is diffeomorphism invariant. Higher derivative terms constructed covariantly from curvature tensors and matter fields do not change this conclusion. The only exceptions are Chern-Simons terms [11], the purely gravitational version being

$$S = \beta \int \Omega_{d+1}\,, \tag{1.2}$$

where the Chern-Simons form is defined as a solution to

$$d\Omega_{d+1} = \mathrm{Tr}\ R^{(d+2)/2}\,. \tag{1.3}$$

Assuming that we have the correct coefficient β in front of (1.2), variation of the action with respect to a diffeomorphism will lead to precise agreement with the gravitational anomaly of the boundary CFT. This anomaly mechanism is the gravitational analog of the Chern-Simons gauge anomaly mechanism explained in [12].

The purpose of this paper, which is in a sense a continuation of our previous work [13], is to flesh out the details of the stress tensor in the presence of gravitational Chern-Simons terms. We will focus primarily on the simplest case of AdS_3. The anomaly leads to a non-conserved stress tensor only when the boundary metric is curved; nevertheless, we will see that the anomaly still makes its presence known even for simple metrics with flat boundary such as global AdS_3. In particular, global AdS_3 acquires a nonzero angular momentum,

$$J = 4\pi\beta\,. \tag{1.4}$$

In the boundary CFT this is to be thought of as a "Casimir momentum" circulating around the boundary. Indeed, in the presence of the anomaly $c_L \neq c_R$, and so the left and right moving zero point momenta do not cancel. The central charges are given by

$$c_L = c_0 + 48\pi\beta, \quad c_R = c_0 - 48\pi\beta\,, \tag{1.5}$$

where c_0 is the central charge in the absence of the Chern-Simons term. We also consider rotating BTZ black holes. The geometry itself is uncorrected by the presence of (1.2), but the expressions for the mass and angular momentum are shifted. Hence the entropy formula, expressed in terms of the mass and angular momentum, is also corrected.

There has recently been much interest in the computation of corrections to the Bekenstein-Hawking area law formula in string theory due to the presence of higher derivatives [14−21, 13]. In certain cases, detailed agreement between microscopic and macroscopic computations has been exhibited to all orders in an inverse charge expansion. However, the success of these comparisons was initially somewhat mysterious, because on the gravity side technical limitations only allow one to include the effect of a certain subset of higher derivative terms. The puzzle was that the omitted terms individually contribute and so will spoil the agreement unless there is some cancellation mechanism.

In [13], building on observations in [15], we showed that this cancellation mechanism follows from symmetries and anomalies, and this further led to a much simplified derivation of the nonzero corrections to the entropy. In particular, to derive the corrections all one needs to know is the coefficient of a particular Chern-Simons term in the action, and the exact value of this coefficient is easily determined from anomalies. Here we will give some more details regarding this analysis.

Most work on the subject of higher derivative corrections has been in the context of black holes with geometry $\mathrm{AdS}_2 \times S^2 \times X$. However, the AdS_2 factor is the "very near horizon" limit of an AdS_3 factor [22], and so one can instead work with $\mathrm{AdS}_3 \times S^2 \times Y$. The latter representation is preferable for our purposes since it makes the full conformal symmetry manifest. In this geometry we can consider anomalies associated with diffeomorphisms in AdS_3 or on S^2. The former gives the gravitational anomaly of the CFT, and so determines $c_L - c_R$. The latter is interpreted as the $SU(2)$ R-symmetry anomaly in the CFT; by supersymmetry this fixes c_L. We can therefore determine c_L and c_R exactly from anomalies [15]. As explained in [13], knowledge of the central charges is sufficient to derive the corrections to the entropy, even when higher derivative corrections are taken into account. The result reproduces the higher derivative entropy formulas in the literature, and explains why they are correct. We also extended the class of examples to include non-BPS states, and states with nonzero angular momentum. Again, it is the powerful constraint of symmetries and anomalies that allow us to make exact statements in these cases.

Gravitational anomalies can show up either in diffeomorphisms, rendering the stress tensor non-conserved, or, if one adopts the vielbein formalism, in local Lorentz transformations, rendering the stress tensor non-symmetric. These are equivalent in the sense that there exists a counterterm that can be added to the action to shift the anomaly from one form to the other [8, 9]. One outcome of our analysis is a simple expression for this counterterm in AdS_3.

In studying anomalies in quantum field theory it is often convenient to think of spacetime as being the boundary of a higher dimensional disk. Various expressions take a simpler form when expressed as integrals over the disk. It is amusing to note that this can be thought of as an indication of holography, for in this context the disk is nothing else than Euclidean AdS_{d+1}.

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This paper is organized as follows. In section 2 we review the holographic stress tensor for Einstein gravity. In section 3 we discuss the nature of the variational principle when higher derivatives are present, and we introduce basic properties of the Chern-Simons term. In section 4 we discuss the effect of anomalies on the stress tensor and on the central charges. In section 5 we compute the stress tensor explicitly, and apply the result to global AdS_3 as well as the BTZ black hole. Finally, in section 6, we discuss how anomalies determine higher derivative corrections to the black hole entropy. Some needed technical results are found in the appendices.

2. Holographic stress tensor in Einstein gravity

In this section we review the derivation of the holographic stress tensor in the case of two-derivative Einstein gravity. This review will also serve to fix conventions and notation. Our curvature conventions follows those in [23].

2.1 The Brown-York stress tensor

We work in $D = d + 1$ euclidean dimensions. It is convenient to adopt Gaussian normal coordinates by foliating the spacetime with d dimensional hypersurfaces labelled by η, and writing the metric as

$$ds^2 = d\eta^2 + g_{ij} dx^i dx^j \,. \tag{2.1}$$

In these coordinates the extrinsic curvature of a fixed η surface reads

$$K_{ij} = \frac{1}{2}\partial_\eta g_{ij} \,. \tag{2.2}$$

The D dimensional Ricci scalar decomposes as

$$R = {}^{(d)}R - (\mathrm{Tr}\, K)^2 - \mathrm{Tr}\, K^2 - 2\partial_\eta \mathrm{Tr}\, K \,, \tag{2.3}$$

where $\mathrm{Tr}\, K = g^{ij} K_{ij}$, and similarly for $\mathrm{Tr}\, K^2$, and ${}^{(d)}R$ denotes the Ricci scalar of the metric g_{ij}.

The D dimensional Einstein-Hilbert action is

$$\begin{aligned} S_{EH} &= \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x \sqrt{g}\,(R - 2\Lambda) \\ &= \frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\, d\eta \sqrt{g}\left({}^{(d)}R + (\mathrm{Tr}\, K)^2 - \mathrm{Tr}\, K^2 - 2\Lambda\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^d x \sqrt{g}\, \mathrm{Tr}\, K \,, \end{aligned} \tag{2.4}$$

where we have taken the boundary $\partial\mathcal{M}$ to be a fixed η surface. The variation of the boundary term contains a contribution $\delta\partial_\eta g_{ij}$ whose presence would spoil the variational principle leading to Einstein's equations. This is rectified by adding to the action the Gibbons-Hawking term

$$S_{GH} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^d x \sqrt{g}\, \mathrm{Tr}\, K \,. \tag{2.5}$$

We now consider the variation of the action with respect to g_{ij}. The variation will consist of two terms: a bulk piece that vanishes when the equations of motion are satisfied, and a boundary piece. Assuming that the equations of motion are satisfied, a simple computation

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gives

$$\delta(S_{EH}+S_{GH}) = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^dx\sqrt{g}\,(K^{ij}-\mathrm{Tr}\,Kg^{ij})\delta g_{ij}\,. \tag{2.6}$$

The stress tensor is defined in terms of the variation as

$$\delta S = \frac{1}{2}\int_{\partial\mathcal{M}} d^dx\sqrt{g}\,T^{ij}\delta g_{ij}\,, \tag{2.7}$$

and so we have

$$T^{ij} = -\frac{1}{8\pi G}(K^{ij}-\mathrm{Tr}\,Kg^{ij})\,, \tag{2.8}$$

which is the result derived by Brown and York [24]. Although we derived this result in the coordinate system (2.1), the result (2.8) is valid in any coordinate system, where g_{ij} is the induced metric on the boundary, and K_{ij} is the extrinsic curvature.

2.2 Asymptotically anti-de Sitter space

With a negative cosmological constant,

$$\Lambda = -\frac{d(d-1)}{\ell^2}\,, \tag{2.9}$$

solutions to Einstein's equations admit the expansion [25].[1]

$$g_{ij} = e^{2\eta/\ell}g^{(0)}_{ij} + g^{(2)}_{ij} + e^{-2\eta/\ell}g^{(4)}_{ij} + \cdots\,. \tag{2.10}$$

Such a solution defines a notion of an asymptotically anti-de Sitter spacetime. We think of the boundary as being at $\eta=\infty$, with metric conformal to $g^{(0)}_{ij}$. We then define the stress tensor in terms of the variation of the action with respect to $g^{(0)}_{ij}$ as in (1.1).

As explained in [12, 26, 1−6], the action and the stress tensor will diverge unless we include additional counterterms that are intrinsic to the boundary [2]. In the case of AdS_3 the counterterm is just the boundary cosmological constant

$$S_{\rm ct} = -\frac{1}{8\pi G\ell}\int_{\partial\mathcal{M}} d^2x\sqrt{g}\,. \tag{2.11}$$

After including the variation of the counterterm, the result for the AdS_3 stress tensor reads

$$T_{ij} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{ij} - g^{kl}_{(0)}g^{(2)}_{kl}\,g^{(0)}_{ij}\right)\,. \tag{2.12}$$

Indices are lowered and raised with $g^{(0)}_{ij}$ and its inverse $g^{ij}_{(0)}$. The stress tensors for higher dimensional spacetimes can be found in the references.

[1] For AdS_{2n+1} with $n\geq 2$ there is also a term linear in η which is related to the conformal anomaly [26, 4] For the remainder of this paper we will focus on AdS_3 where this term is absent [4], and so we neglect it henceforth.

[2] The counterterms are also required if we generalize the argument motivating the Gibbons-Hawking term (2.5) to take into account that, on an AdS-boundary, it is the *conformal* metric that must be kept fixed [6].

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3. General aspects of higher derivative theories

In this section we discuss the addition of higher derivative terms to the action, either built out of curvature invariants, or as Chern-Simons terms.

3.1 The variational principle for higher derivative actions

The variation of the action with respect to the metric generally produces a boundary term, as illustrated for Einstein gravity after (2.5). For an action containing up to n derivatives, the boundary term will typically involve $(\partial_\eta)^k \delta g_{ij}$ with $k = 0, 1, \ldots n-1$. The standard variational principle seeks an extremum among configurations with fixed boundary values of the field, i.e. it takes $\delta g_{ij} = 0$. The terms with $k \geq 0$ show that solutions to the bulk equations of motion fail to extremize the action, and so they undercut the standard variational principle. In a generic higher derivative theory one might therefore modify the variational principle by imposing additional boundary conditions, such as $(\partial_\eta)^k \delta g_{ij} = 0$. However, this approach clashes with the AdS/CFT correspondence, since it would imply the existence of $n-1$ additional "stress tensors" with no obvious analog in the CFT. Instead, AdS/CFT implies that we should work with the unmodified variational principle; there are two ways in which its validity can be restored.

The first possibility is that there might exist a generalized Gibbons-Hawking term whose variation precisely cancels the $(\partial_\eta)^k \delta g_{ij}$ terms for $k \geq 1$. This is a stringent requirement which is met only for very special choices of bulk actions. For instance, given a general four-derivative action constructed from a linear combination of R^2, $R^{\mu\nu}R_{\mu\nu}$, and $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, the existence of a suitable Gibbons-Hawking term fixes the relative coefficients to be those of the dimensionally continued $D = 4$ Euler invariant

$$S_{\mathrm{Euler}} = c_E \left(R^2 - 4 R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right) . \tag{3.1}$$

Generically no generalized Gibbons-Hawking term exists and so the variational principle requires the specification of all $(\partial_\eta)^k \delta g_{ij}$ on the boundary, it does not allow them to fluctuate.

The second possibility is to exploit the fact that the boundary is a surface at infinity, and define falloff conditions such that the coefficients of the unwanted terms $(\partial_\eta)^k \delta g_{ij}$ for $k \geq 1$ vanish at infinity. This is what happens for the higher derivative Chern-Simons terms we consider here, using the Fefferman-Graham expansion (2.10). In terms of (2.10), the actual statement we will need is that the variation of the action only involves $\delta g_{ij}^{(0)}$ and not $\delta g_{ij}^{(2n)}$ for $n > 0$. A natural question, which we do not address here, is to what extent the form of more complicated higher derivative bulk actions is constrained by imposing this condition.

3.2 Gravitational Chern-Simons terms

We now turn our attention specifically to Chern-Simons terms and their associated anomalies. Our conventions will follow those in [10], which is a helpful reference for what follows; see also [27].

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We define the connection 1-form as

$$\Gamma^{\alpha}{}_{\beta} = \Gamma^{\alpha}_{\beta\mu} dx^{\mu} , \tag{3.2}$$

where $\Gamma^{\alpha}_{\beta\mu}$ are the usual Christoffel symbols. The standard definition of the Riemann tensor is then equivalent to defining the curvature 2-form as

$$R^{\alpha}{}_{\beta} = d\Gamma^{\alpha}{}_{\beta} + \Gamma^{\alpha}{}_{\gamma} \wedge \Gamma^{\gamma}{}_{\beta} , \tag{3.3}$$

or, using a matrix notation, as

$$R = d\Gamma + \Gamma \wedge \Gamma . \tag{3.4}$$

Upon writing $R^{\alpha}{}_{\beta} = \frac{1}{2} R^{\alpha}_{\beta\mu\nu} dx^{\mu} \wedge dx^{\nu}$ we recover the standard component definition of the Riemann tensor. To further simplify notation, we will often suppress the explicit wedge products among forms.

Under the infinitesimal diffeomorphism $x^{\mu} \to x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$ the metric, connection, and curvature transform as[3]

$$\begin{aligned} \delta_{\xi} g_{\mu\nu} &= v_{\mu\nu} + v_{\nu\mu} , \\ \delta_{\xi} \Gamma &= dv + [\Gamma, v] , \qquad \delta_{\xi} R = [R, v] , \end{aligned} \tag{3.5}$$

where

$$v^{\alpha}{}_{\beta} = \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} . \tag{3.6}$$

The Chern-Simons 3-form

$$\Omega_3(\Gamma) = \mathrm{Tr}(\Gamma d\Gamma + \frac{2}{3}\Gamma^3) \tag{3.7}$$

is central to our applications. It has two key properties: first, its exterior derivative is a symmetric polynomial

$$d\Omega_3(\Gamma) = \mathrm{Tr}(R^2) , \tag{3.8}$$

and second, under a diffeomorphism it varies by a total derivative

$$\delta_{\xi} \Omega_3(\Gamma) = d\, \mathrm{Tr}(v d\Gamma) . \tag{3.9}$$

Due to the latter property a term in the action of the form

$$S_{CS}(\Gamma) = \int_{\mathcal{M}} \Omega_3(\Gamma) \tag{3.10}$$

transforms under diffeomorphism by a boundary term,

$$\delta_{\xi} S_{CS}(\Gamma) = \int_{\partial \mathcal{M}} \mathrm{Tr}(v d\Gamma) . \tag{3.11}$$

Thus, diffeomorphism invariance of the bulk theory is preserved by (3.10) in the sense that the equations of motion remain covariant, even though the full action is not invariant. The holographic interpretation will be that the dual theory on the boundary suffers a gravitational anomaly.

[3] To keep formulae simple and to conform with [10] we do not shift the argument of the functions explicitly. Doing so would in any case yield the same variation of the action as long as the location of the boundary is kept fixed. The full change, needed later, amounts to defining $v^{\alpha}{}_{\beta}$ using a covariant derivative.

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It is instructive to consider also an alternative formalism that preserves diffeomorphism invariance at the expense of local Lorentz invariance. Now the geometry is represented by the vielbein $e^a = e^a{}_\mu dx^\mu$ and, rather than the connection one-form (3.2), we introduce the spin-connection $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$ determined by Cartan's structure equation

$$de^a + \omega^a{}_b e^b = 0\,. \tag{3.12}$$

The curvature 2-form is then

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b\,. \tag{3.13}$$

Under an infinitesimal local Lorentz transformation parameterized by the matrix $\Theta^a{}_b = -\Theta_b{}^a$ we have

$$\begin{aligned} \delta_\Theta e &= -\Theta e\,, \\ \delta_\Theta \omega &= d\Theta + [\omega, \Theta]\,, \qquad \delta_\Theta R = [R, \Theta]\,, \end{aligned} \tag{3.14}$$

where Lorentz indices are implied. Since e, ω, and R are differential forms they are invariant under general coordinate transformations.

In the vielbein formalism we define the Chern-Simons 3-form

$$\Omega_3(\omega) = \mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right)\,, \tag{3.15}$$

obeying

$$\begin{aligned} d\Omega_3(\omega) &= \mathrm{Tr}(R^2)\,, \\ \delta\Omega_3(\omega) &= d\,\mathrm{Tr}(\Theta d\omega)\,. \end{aligned} \tag{3.16}$$

Again, a Chern-Simons term in the action,

$$S_{CS}(\omega) = \int_{\mathcal{M}} \Omega_3(\omega)\,, \tag{3.17}$$

has a variation that localizes on the boundary

$$\delta S_{CS}(\omega) = \int_{\partial\mathcal{M}} \mathrm{Tr}(\Theta d\omega)\,. \tag{3.18}$$

Thus the bulk theory remains Lorentz invariant in the presence of the term (3.17), but the dual boundary theory does not. This is the alternative manifestation of the gravitational anomaly that we wanted to exhibit.

For tensors, the transcription between the two alternate formalisms is the obvious one. For example, the curvature two forms (3.3) and (3.13) are related by $R^\alpha{}_\beta = e^\alpha{}_a R^a{}_b e^b{}_\beta$. However, the relation between connection one-forms and the spin connections contains an inhomogeneous term

$$\Gamma^\alpha{}_\beta = e^\alpha{}_a \omega^a{}_b e^b{}_\beta + e^\alpha{}_a \partial_\mu e^a{}_\beta dx^\mu\,. \tag{3.19}$$

This explains how the two alternate forms of the Chern-Simons action (3.10) and (3.17) manage to preserve different symmetries: they are not equal. It is convenient to write (3.19) in a more condensed fashion as

$$\Gamma = e^{-1}\omega e + e^{-1} de\,. \tag{3.20}$$

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Here and in the next three formulas e is interpreted as a matrix valued 0-form, *not* as a 1-form as it is elsewhere; we hope this will not cause confusion. In this notation, the difference between the two forms of the Chern-Simons action is

$$\Delta S_{CS} \equiv S_{CS}(\Gamma) - S_{CS}(\omega) = -\frac{1}{3}\int_{\mathcal{M}} \mathrm{Tr}(e^{-1}de)^3 + \int_{\partial\mathcal{M}} \mathrm{Tr}(\omega dee^{-1})\,. \tag{3.21}$$

By construction, ΔS_{CS} transforms under general coordinate and local Lorentz transformations as

$$\begin{aligned} \delta_\xi \Delta S_{CS} &= \int_{\partial\mathcal{M}} \mathrm{Tr}(v d\Gamma)\,, \\ \delta_\Theta \Delta S_{CS} &= -\int_{\partial\mathcal{M}} \mathrm{Tr}(\Theta d\omega)\,. \end{aligned} \tag{3.22}$$

Thus, the addition (or subtraction) of ΔS_{CS} to the action transforms between the two different forms of the anomaly. Since the general variation of ΔS_{CS} localizes on the boundary

$$\delta\Delta S_{CS} = -\int_{\partial\mathcal{M}} \mathrm{Tr}(\delta ee^{-1}dee^{-1}dee^{-1}) + \int_{\partial\mathcal{M}} \delta\,\mathrm{Tr}(\omega dee^{-1})\,, \tag{3.23}$$

this term can be interpreted as intrinsic to the boundary theory. This means that the two forms of the gravitational anomaly are equivalent.

4. Holographic gravitational anomalies

In this section we discuss the effects of gravitational anomalies on the boundary stress tensor, and the interpretation of these as shifts in the central charges.

4.1 Anomalous conservation laws

Conventionally, a general three dimensional action for gravity coupled to matter is constructed by forming invariant terms from curvature tensors, matter fields, and their covariant derivatives.[4] Classically, such an action would be invariant under general coordinate transformations[5] and local Lorentz transformations. When a Chern-Simons term is added, these symmetries may be violated by a nonzero boundary variation as was determined explicitly in the previous section. This manifests itself in unusual properties of the boundary stress tensor which we discuss in the following.

We first consider the diffeomorphism anomaly, i.e. we add the term $\beta S_{CS}(\Gamma)$ to the action. Quite generally, by the definition of the stress tensor, we can write the variation of an action due to a general coordinate transformation as

$$\delta_\xi S = \frac{1}{2}\int_{\partial\mathcal{M}} d^2x\sqrt{g}\; T^{ij}\delta g_{ij} = \int_{\partial\mathcal{M}} d^2x\sqrt{g}\; T^{ij}\nabla_i\xi_j = -\int_{\partial\mathcal{M}} d^2x\sqrt{g}\;\nabla_i T^{ij}\xi_j\,. \tag{4.1}$$

[4]In general, additional boundary terms are needed for a well-defined variational principle, as discussed in section 3.1. Furthermore, as mentioned in section 2.2, local counter-terms on the boundary are needed to render the action finite.

[5]For a manifold with boundary we should demand that the coordinate transformation does not shift the location of the boundary.

Since the anomaly arises exclusively from the Chern-Simons term we can compare this with (3.11) and find the anomalous divergence of the stress tensor

$$\nabla_i T^{ij} = g^{ij}\epsilon^{kl}\partial_k\partial_m\Gamma^m_{il}\,. \tag{4.2}$$

Next, we consider the Lorentz anomaly, i.e. we add $\beta S_{CS}(\omega)$ to the action. In the vielbein formalism the variation of the action is

$$\delta_\Theta S = \int_{\partial\mathcal{M}} d^2x\; e\;\delta e^a{}_\mu T_{ab}e^{b\mu} = -\int_{\partial\mathcal{M}} d^2x\; e\;\Theta^{ab}T_{ab}\,. \tag{4.3}$$

Comparing this with the variation (3.18) we learn that the stress tensor picks up an anti- symmetric contribution:

$$T_{ab} - T_{ba} = 2\beta\;{}^\star R_{ab}\,, \tag{4.4}$$

where ${}^\star R_{ab}$ is the Hodge dual of the boundary curvature 2-form, i.e. a 0-form.

Anomalies in general coordinate transformations thus manifest themselves in non-conservation of the stress tensor, while anomalies in local Lorentz transformations show up in the asymmetric part of the stress tensor. These are not really independent anomalies since we exhibited in (3.21) a term that can be added to the action with an appropriate coefficient so as to cancel one or the other of the anomalies. Therefore, the invariant statement is that there is an anomaly in *either* general coordinate or local Lorentz transformations. We further remark that this anomaly shifting counterterm ΔS_{CS} can be thought of as being defined on the boundary, since its variation is strictly localized there. All of this was known from the early days of gravitational anomalies [8],[9], although the completely explicit form of the anomaly shifting counterterm was not written down, as far as we are aware.

4.2 Anti-de Sitter spacetime

The results of the last subsection apply to any three dimensional geometry, including AdS_3. As we have noted, in AdS_3 one takes the conformal boundary metric to be $g^{(0)}_{ij}$. Our previous results for the variation of the action apply with the boundary metric taken to be $g^{(0)}_{ij}$.

In the context of the AdS/CFT correspondence we can compare our bulk variations to the anomalous variations of the boundary CFT. If the boundary theory has central charges c_L and c_R then under a general coordinate transformation

$$\delta_\xi S = \frac{c_L - c_R}{96\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}(v d\Gamma)\,. \tag{4.5}$$

This assumes we choose to set the local Lorentz anomaly to zero. Equivalently, if we choose to set the general coordinate anomaly to zero, then the local Lorentz anomaly is

$$\delta_\Theta S = \frac{c_L - c_R}{96\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}(\Theta d\omega)\,. \tag{4.6}$$

In either case, we learn that a bulk action with Chern-Simons term βS_{CS} corresponds to a theory with

$$c_L - c_R = 96\pi\beta\,. \tag{4.7}$$

The Chern-Simons term is maximally chiral, *i.e.* it treats left and right oppositely. Therefore, the shifts in left and right central charges must be equal in magnitude, but of opposite sign. So we can write (4.7) as

$$c_L = c_0 + 48\pi\beta\,, \quad c_R = c_0 - 48\pi\beta\,, \tag{4.8}$$

where c_0 is the central charge in the absence of the Chern-Simons term.

4.3 Gravitational anomaly for $\mathbf{AdS}_3 \times S^p$

Now consider a theory admitting $\mathrm{AdS}_3 \times S^p$ as a solution, with the sphere supported by p-form flux

$$-\frac{1}{2\pi}\int_{S^p} F^{(p)} = q\,. \tag{4.9}$$

We can consider gravitational anomalies associated with transformations on the sphere.

For definiteness, we phrase the anomaly in terms of local Lorentz transformations. Consider the following deformation of $\mathrm{AdS}_3 \times S^p$

$$\begin{aligned} ds^2 &= ds_3^2(x) + \sum_{m=1}^{p} (e^m)^2\,, \\ e^m &= dy^m - A^m_{\ n}(x) y^n\,, \\ \sum_{m=1}^{p} (y^m)^2 &= R_{S^p}^2\,, \end{aligned} \tag{4.10}$$

where x^μ and y^m denote the AdS_3 and S^p coordinates, respectively. $A^m_{\ n}$ can be identified with the spin connection on the sphere: $A^m_{\ n} = \omega^m_{\ n}$.

Suppose that in this theory there exists a Chern-Simons term of the form

$$S_{CS} = \gamma \int F^{(p)} \wedge \Omega_3\,. \tag{4.11}$$

We can reduce this term to $D = 3$ by integrating over S^p, yielding

$$S_{CS} = \beta \int \Omega_3(\omega) + \beta \int \Omega_3(A)\,, \tag{4.12}$$

with $\beta = -2\pi\gamma q$. We have assumed that $F^{(p)}$ only has components on S^p, and so in (4.12) we only get the contribution of the spin connection when its 1-form index is in AdS_3.

The first term in (4.12) is the gravitational Chern-Simons term that we have studied in the previous sections. The second term is best interpreted as an $SO(p+1)$ Yang-Mills Chern-Simons term, where the $SO(p+1)$ gauge invariance corresponds to isometries of the sphere. The presence of the Chern-Simons term means that there is a nonzero anomalous boundary contribution associated with $SO(p+1)$ gauge transformations, $\delta A = d\Lambda + [A, \Lambda]$,

$$\delta S = \beta \int_{\partial\mathcal{M}} \mathrm{Tr}(\Lambda dA)\,. \tag{4.13}$$

In the context of AdS/CFT, (4.13) is interpreted in the CFT as a contribution to the R-symmetry anomaly. Well known cases are the D1-D5 system, described by $p = 3$ and corresponding $SO(4) \cong SU(2) \times SU(2)$ R-symmetry, and M-theory on CY_3 with wrapped M5-branes, corresponding to $p = 2$ and $SU(2)$ R-symmetry. The latter example is especially relevant in the context of higher derivative corrections, as we discuss later.

In contexts where AdS_3 arises as the decoupled geometry near some branes, transformations on the sphere amount to rotations of the vectors normal to the brane worldvolume. In this case A is interpreted as the connection of the normal bundle, and the associated gravitational anomaly is the normal bundle anomaly discussed in [28].

We should emphasize that the above Yang-Mills Chern-Simons term arising from (4.11) is a *correction*; there is typically such a term present even starting from the two-derivative Einstein action, although its derivation can be somewhat subtle [15]. Finally, we remark that corrections to anomalies in the AdS_5 context have been discussed in [29][30][31][32][33]-[34]

5. Holographic stress tensor in the presence of Chern-Simons terms

In this section we compute the contribution of the gravitational Chern-Simons term to the boundary stress tensor in an asymptotically AdS_3 spacetime and apply the result to the case where the bulk geometry is either global AdS_3 or the BTZ black hole.

5.1 Derivation

We want to find the contribution of

$$S_{CS}(\Gamma) = \int_{\mathcal{M}} \Omega_3(\Gamma) = \int_{\mathcal{M}} \left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3 \right) \quad , \tag{5.1}$$

to the stress tensor. To do this we must work out the change in (5.1) due to a variation of the metric around a solution of the equations of motion. A useful first step is to vary the connection. This yields

$$\delta S_{CS}(\Gamma) = 2\int_{\mathcal{M}} \mathrm{Tr}(\delta\Gamma \wedge R) - \int_{\partial\mathcal{M}} \mathrm{Tr}(\Gamma \wedge \delta\Gamma)\,. \tag{5.2}$$

The curvature two form was defined in (3.4).

Next, we write the variation of the connection in terms of the underlying metric and simplify the resulting expression by introducing Gaussian normal coordinates (2.1). Starting from the first term in (5.2) this procedure gives

$$2\int_{\mathcal{M}} \mathrm{Tr}(\delta\Gamma \wedge R) = -\int_{\mathcal{M}} d^3x\sqrt{g}\,\delta g_{\gamma\rho}(\nabla_\beta R^{\beta\rho}{}_{\mu\nu})\epsilon^{\gamma\mu\nu} + 2\int_{\partial\mathcal{M}} d^2x\sqrt{g}\ \delta g_{ij} R^{\eta j}{}_{\eta k}\epsilon^{i\eta k}\,, \tag{5.3}$$

as detailed in appendix B. The bulk term gives the correction to the equations of motion. However, this term vanishes in the important case of a space whose curvature is covariantly constant. So, for example, the metrics of pure AdS_3 and BTZ black holes are uncorrected by $S_{CS}(\Gamma)$. The boundary term in (5.3) would seem to contribute to the boundary stress tensor; however, the AdS$_3$ boundary is at large η, where the Fefferman-Graham expansion (2.10) applies. As we detail in appendix A, $R^{\eta j}{}_{\eta k}$ is proportional to δ^j_k in this limit. This means the last term in (5.3) vanishes.

At this point we have shown that the first term in (5.2) does not contribute to the boundary stress tensor. Before evaluating the second term in (5.2) explicitly, we can determine its large η behavior using the Fefferman-Graham expansion (2.10) and find, confusingly, that it diverges at the boundary. Happily, the explicit computation (in appendix B) shows that the leading terms for large η cancel, leaving a finite answer. Due to this cancellation, no boundary counterterms are needed beyond those required by Einstein gravity (discussed in section 2.2). More importantly, it shows that the total boundary variation of $S_{CS}(\Gamma)$

$$\delta S_{CS}(\Gamma) = \text{bulk} - \int_{\partial\mathcal{M}} \text{Tr}(\Gamma \wedge \delta\Gamma) = \text{bulk} + \frac{1}{2}\int_{\partial\mathcal{M}} d^2x \sqrt{g^{(0)}} T^{ij} \delta g^{(0)}_{ij} \,, \tag{5.4}$$

comes purely from the leading asymptotic metric $\delta g^{(0)}$. As discussed in section 3.1, this is a requirement for the existence of a good variational principle.

The explicit result for the contribution of the Chern-Simons term to the stress tensor is of the form

$$T^{ij} = t^{ij} + X^{ij} \,, \tag{5.5}$$

where

$$t^{ij} = \frac{2}{\ell^2}\left(g^{ik}_{(2)}\epsilon^{lj} + g^{jk}_{(2)}\epsilon^{li}\right) g^{(0)}_{kl} \,, \tag{5.6}$$

is the contribution from the extrinsic part of the connection, i.e. the Γ^{η}_{mn} and $\Gamma^{m}_{\eta n}$ components, and X^{ij} is defined through

$$\frac{1}{2}\int d^2x \sqrt{g^{(0)}} X^{ij} \delta g^{(0)}_{ij} = -\int d^2x \sqrt{g^{(0)}}\, \Gamma^i_{jk} \delta\Gamma^j_{il} \epsilon^{kl} \,, \tag{5.7}$$

where the connection is formed from the boundary metric $g^{(0)}_{ij}$. Since t^{ij} depends on $g^{(2)}_{ij}$, it is sensitive to the precise geometry of the bulk space-time, rather than just its conformal structure at infinity. We can think of this as a dependence on the state of the theory. In contrast, X^{ij} is formed from $g^{(0)}_{ij}$ alone, and defined so that it vanishes when the connection constructed from $g^{(0)}_{ij}$ is trivial. In explicit computations with a fixed conformal structure X^{ij} appears as a background constant. Indeed, in most examples, including the rotating BTZ black hole in standard coordinates, and pure AdS$_3$ expressed in global or Poincaré coordinates, we have $X^{ij} = 0$, and hence $T^{ij} = t^{ij}$. The explicit result for the stress tensor in these geometries is given in the next subsection.

Let us also comment on the trace anomaly in the presence of a Chern-Simons term. It is manifest from (5.6) that the trace of the state-dependent contribution vanishes. Also, for a rigid (position independent) Weyl transformation $\delta g^{(0)}_{ij} = \delta\sigma g^{(0)}_{ij}$ we have $\delta\Gamma^j_{kl} = 0$ and so X^{ij} vanishes upon integration. Thus, if we match the trace of the stress tensor to the usual covariant form of the trace anomaly

$$T^i_i = -\frac{c_0}{12} R \,, \tag{5.8}$$

we find that c_0 is uncorrected by the Chern-Simons term. In the present context the boundary theory is not diffeomorphism invariant so there may be additional, non-covariant, terms on the right hand side of (5.8). However, such terms vanish upon integration.

5.2 Examples: global AdS$_3$ and the BTZ black hole

The metric of the rotating BTZ black hole is [35]

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 (d\phi - \frac{r_+ r_-}{\ell r^2} dt)^2 . \tag{5.9}$$

In this subsection we work in Lorentzian signature in order to avoid awkward imaginary angular momenta. We define

$$m = \frac{r_+^2 + r_-^2}{8G_3 \ell^2} , \quad j = \frac{2r_+ r_-}{4G_3 \ell} . \tag{5.10}$$

In this parametrization global AdS$_3$ is the special case $m = -\frac{1}{8G_3}$, $j = 0$, and AdS$_3$ in Poincaré coordinates corresponds to $m = j = 0$.

In ordinary Einstein gravity m and j are identified with the mass and angular momentum of the black hole. We wish to see how this is modified due to the Chern-Simons term

$$S_{CS}(\Gamma) = \beta \int_{\mathcal{M}} \Omega_3(\Gamma) . \tag{5.11}$$

Transforming (5.9) into Gaussian normal coordinates using

$$\left(\frac{r}{\ell}\right)^2 = e^{2\eta/\ell} + 4G_3 m + \cdots , \tag{5.12}$$

and expanding for large η we have

$$ds^2 = d\eta^2 + e^{2\eta/\ell}(-dt^2 + \ell^2 d\phi^2) + (4G_3 m dt^2 + 4G_3 m \ell^2 d\phi^2 - 8G_3 j dt d\phi) + \cdots , \tag{5.13}$$

The expressions in brackets in this equation are identified with the components $g_{ij}^{(0)}$ and $g_{ij}^{(2)}$ of the Fefferman-Graham expansion (2.10). The stress tensor, including the contribution (2.12) from the Einstein-Hilbert term, is

$$T_{ij} = \frac{1}{8\pi G_3 \ell} \left[g_{ij}^{(2)} - g_{ij}^{(0)} g_{kl}^{(2)} g_{(0)}^{kl} \right] + \frac{2\beta}{\ell^2} \left[g_{ik}^{(2)} \epsilon_{lj} g_{(0)}^{kl} + g_{jk}^{(2)} \epsilon_{li} g_{(0)}^{kl} \right] . \tag{5.14}$$

Inserting $g^{(0)}$ and $g^{(2)}$ from (5.13), and taking the orientation $\epsilon_{t\phi} = -\ell$, we find

$$\begin{aligned} T_{tt} &= \frac{m}{2\pi\ell} - \frac{16\beta G_3 j}{\ell^3} , \\ T_{\phi\phi} &= \frac{m\ell}{2\pi} - \frac{16\beta G_3 j}{\ell} , \\ T_{t\phi} &= -\frac{j}{2\pi\ell} + \frac{16 G_3 \beta m}{\ell} . \end{aligned} \tag{5.15}$$

Therefore, the corrected formulas for the mass and angular momentum of the BTZ black hole are

$$\begin{aligned} M &= 2\pi\ell T_{tt} = m - \frac{32\pi\beta G_3 j}{\ell^2}\,, \\ J &= -2\pi\ell T_{t\phi} = j - 32\pi\beta G_3 m\,. \end{aligned} \tag{5.16}$$

Alternatively, we can express these results in terms of the Virasoro generators as

$$\begin{aligned} L_0 - \frac{c_L}{24} &= \frac{M\ell - J}{2} = \left(1 + \frac{32\pi\beta G_3}{\ell}\right)\frac{m\ell - j}{2} \\ \tilde{L}_0 - \frac{c_R}{24} &= \frac{M\ell + J}{2} = \left(1 - \frac{32\pi\beta G_3}{\ell}\right)\frac{m\ell + j}{2}\,. \end{aligned} \tag{5.17}$$

An important special case is global AdS_3 where $m = -\frac{1}{8G_3}$ and $j = 0$ and so

$$M_{AdS_3} = -\frac{1}{8G_3}, \quad J_{AdS_3} = 4\pi\beta\,. \tag{5.18}$$

In particular, global AdS_3 carries a non-vanishing angular momentum. For the dual CFT perspective on this result, recall that global AdS_3 corresponds to the NS-NS vacuum which has $L_0 = \tilde{L}_0 = 0$, and thus

$$c_L = \frac{3\ell}{2G} + 48\pi\beta\,, \quad c_R = \frac{3\ell}{2G} - 48\pi\beta\,, \tag{5.19}$$

which gives the correction to the usual Brown-Henneaux result for the central charge. With this in hand, we can reexpress (5.18) as

$$M_{AdS_3}\ell = -\frac{c_L + c_R}{24}\,, \quad J_{AdS_3} = \frac{c_L - c_R}{24}\,. \tag{5.20}$$

This shows that the ground state angular momentum is due to the asymmetry of the central charges, which implies a non-cancellation between the left and right moving zero point momenta.

6. Application to black hole entropy in string theory

In [13], building on arguments in [15], we showed that gravitational anomalies can be used to compute higher derivative corrections to black hole entropy. This explains the success of recent computations which take into account a subset of higher derivative terms and find agreement with microscopic entropy counting. As explained in [13], these terms include the effects of anomalies, and this, along with symmetries, is enough to determine the corrections to the entropy. In this section we review this argument.

Our argument applies to black holes which have a BTZ factor. To make contact with other papers in the literature which deal with near horizon AdS_2 geometries, we note that if we Kaluza-Klein reduce a BTZ black hole along the horizon direction we recover a near-horizon AdS_2 factor [22]. However, this reduction obscures some of the symmetries, namely the existence of left and right moving Virasoro algebras, and so we will stick to the BTZ description.

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In [13] we showed that in a general higher derivative theory of gravity the entropy of a BTZ black hole is

$$S = 2\pi\left[\sqrt{\frac{c_L h_L}{6}} + \sqrt{\frac{c_R h_R}{6}}\right], \tag{6.1}$$

with

$$h_L = \frac{M\ell - J}{2}, \quad h_R = \frac{M\ell + J}{2}. \tag{6.2}$$

In a diffeomorphism invariant theory (6.1) is equivalent to Wald's generalized entropy formula [36], as discussed in [37, 13]. (6.1) is actually more general (in the case of BTZ black holes) in that it also applies in the presence of Chern-Simons terms. (6.1) is just the leading saddle point contribution to the entropy. More generally, one has to perform an inverse Laplace transform to convert a partition function to a density of states, but we omit this here; for more details see the discussion in the last section of [13].

So to compute black hole entropy we just need to determine the central charges $c_{L,R}$. In a general theory this requires knowing the full action and carrying out the c-extremization procedure discussed in [13]. But in string theory examples with enough supersymmetry, the central charges are related to anomalies, and these can be computed exactly purely from knowledge of Chern-Simons terms.

A case of particular interest corresponds to wrapping M5-branes on a 4-cycle P_0 of a Calabi-Yau threefold. This gives rise to a magnetic string in $D = 5$, described at low energies by a CFT with $(4, 0)$ supersymmetry. In this CFT, the gravitational anomaly determines $c_L - c_R$, and the anomaly with respect to the leftmoving $SU(2)$ R-symmetry (which always exists as part of the superconformal algebra) determines c_L. For us, the key point is that from a supergravity point of view these anomalies are determined from Chern-Simons terms. Furthermore, in string theory the coefficients in front of Chern-Simons terms are known exactly, since they are required for anomaly cancellation.

For the $D = 5$ case just mentioned, the relevant Chern-Simons term is

$$S = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \frac{c_2 \cdot P_0}{48} \int F^{(2)} \wedge \Omega_3, \tag{6.3}$$

where $c_2 \cdot P_0$ denotes the second Chern class of the $P_0 \subset CY_3$, and $F^{(2)}$ has flux corresponding to charge q, as in (4.9). The magnetic string supports an $\mathrm{AdS}_3 \times S^2$ solution,[6] so we can follow the analysis leading to (4.12), with

$$\beta = -\frac{1}{2}\left(\frac{1}{2\pi}\right)\frac{c_2 \cdot q}{48}. \tag{6.4}$$

From (4.7) we learn that

$$c_L - c_R = 96\pi\beta = -\frac{1}{2}c_2 \cdot q. \tag{6.5}$$

The leftmoving supersymmetry implies that the $SU(2)$ current algebra has level $k = c_L/6$, and associated anomaly

$$\delta S = -\frac{c_L}{96\pi}\int \mathrm{Tr}(\Lambda dA). \tag{6.6}$$

[6]The scalars are fixed by the 5D attractor mechanism [38–40].

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Comparing with (4.13) we find

$$\Delta c_L = 96\pi\beta = \frac{1}{2}c_2 \cdot q\,. \tag{6.7}$$

Here we have indicated that (6.7) just gives the correction to c_L; there is also a term coming from the Chern-Simons term in the lowest order supergravity theory. Altogether one gets

$$c_L = C_{IJK}q^I q^J q^K + \frac{1}{2}c_2 \cdot q, \quad c_R C_{IJK}q^I q^J q^K + c_2 \cdot q\,. \tag{6.8}$$

As we have stressed, this result is exact. Any correction would imply a noncancellation of anomalies in the theory of an M5-brane in M-theory, and so would lead to an inconsistency.

Inserting this result into (6.1) we get a result for black hole entropy including higher order derivative corrections. This result is in agreement with the much more involved derivation in [16], which requires knowing the full supergravity action at the level of R^2 terms in $D = 4$. The only assumptions we needed to make was that there exists an $\mathrm{AdS}_3 \times S^2$ vacuum corresponding to our microscopic system, and that the effective supergravity theory respects the $(4, 0)$ symmetry. If the symmetry assumption is false it will represent a breakdown of the AdS/CFT correspondence.

In other approaches it is unclear why even higher derivative corrections, of R^4 type and beyond, don't destroy the agreement. Such terms are certainly present, and do individually lead to unwanted corrections. The point is that the $(4, 0)$ symmetry implies that the sum of all such contributions must vanish. Any non-vanishing total result would, by symmetry, lead to a shift in the anomalies, and this would lead to an inconsistency as we have discussed.

We emphasize that while we have used supersymmetry of the underlying theory, we have not demanded that our black hole preserves supersymmetry. Thus the entropy formula (6.1) applies even to non-supersymmetric BTZ black holes. These correspond to the case when h_L is nonvanishing.

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A. Gaussian normal coordinates and the Fefferman-Graham expansion

In this appendix we collect a number of useful formulae pertaining to the use of Gaussian normal coordinates and the Fefferman-Graham expansion. The formulae are valid in arbitrary dimension.

We define the Gaussian normal coordinates by the line element

$$ds^2 = d\eta^2 + g_{ij}dx^i dx^j\,. \tag{A.1}$$

In these coordinates the extrinsic curvature of a surface at fixed η is given by $K_{ij} = \frac{1}{2}\partial_\eta g_{ij}$. The nonvanishing Christoffel symbols are

$$\Gamma^{\eta}_{ij} = -K_{ij}\,, \quad \Gamma^{i}_{\eta j} = K^{i}_{j}\,, \tag{A.2}$$

as well as the Γ^k_{ij} which is the same from the bulk and the boundary point of view.

The components of the Riemann tensor are

$$\begin{aligned} R^{i}{}_{jk\eta} &= {}^{(d)}\nabla^i K_{kj} - {}^{(d)}\nabla_j K^i_k\,, \\ R^{i}{}_{\eta j\eta} &= -K^k_j K^i_k - \partial_\eta K^i_j\,, \\ R^{i}{}_{jkl} &= {}^{(d)}R^{i}{}_{jkl} + K_{kj}K^i_l - K_{lj}K^i_k\,. \end{aligned} \tag{A.3}$$

By contraction, we find the Ricci tensor

$$\begin{aligned} R_{\eta\eta} &= -\operatorname{Tr} K^2 - \partial_\eta \operatorname{Tr} K\,, \\ R_{\eta i} &= {}^{(d)}\nabla^j K_{ij} - {}^{(d)}\nabla_i \operatorname{Tr} K\,, \\ & \quad R^{i}{}_{j}R^{i}{}_{j} - K^{i}{}_{j} \operatorname{Tr} K - \partial_\eta K^{i}{}_{j}\,, \end{aligned} \tag{A.4}$$

and the Ricci scalar

$$R = {}^{(d)}R - (\operatorname{Tr} K)^2 - \operatorname{Tr} K^2 - 2\partial_\eta \operatorname{Tr} K\,. \tag{A.5}$$

Spaces that are asymptotically AdS_{d+1} allow the Fefferman-Graham expansion[7]

$$g_{ij} = e^{2\eta/\ell} g^{(0)}_{ij} + g^{(2)}_{ij} + \dots\,. \tag{A.6}$$

The corresponding inverse expansion is

$$g^{ij} = e^{-2\eta/\ell} g^{ij}_{(0)} - e^{-4\eta/\ell} g^{ij}_{(2)} + \dots\,, \tag{A.7}$$

with the understanding that indices on $g_{(2)}$ are raised and lowered using $g_{(0)}$. Some useful formulae for the expansion of the extrinsic curvature are

$$\begin{aligned} K_{ij} &= \frac{1}{\ell} e^{2\eta/\ell} g^{(0)}_{ij} + 0 + \cdots\,, \\ K^i_j &= \frac{1}{\ell}\delta^i_j - \frac{1}{\ell} e^{-2\eta/\ell} g^{ij}_{(2)} g^{(0)}_{lj} + \cdots\,. \end{aligned} \tag{A.8}$$

The expansions for the Christoffel symbols with one extrinsic index follows from (A.2). The leading term in the symbol Γ^k_{ij} is simply formed from $g_{(0)}$ and is η independent.

The components of the Riemann curvature expand as

$$\begin{aligned} R^i_{\eta k\eta} &= -\frac{1}{\ell^2}\delta^i_k + 0 + \cdots\,, \\ R^i_{jk\eta} &= -\frac{1}{\ell} e^{-2\eta/\ell}\left[g^{(0)}_{jm}\, {}^{(d)}\nabla^i g^{ml}_{(2)} - {}^{(d)}\nabla_j g^{il}_{(2)}\right] g^{(0)}_{lk} + \cdots\,, \\ R^i_{jkl} &= \frac{1}{\ell^2} e^{2\eta/\ell}\left(g^{(0)}_{kj}\delta^i_l - g^{(0)}_{lj}\delta^i_k\right) + \left[{}^{(d)}R^i_{jkl} - \frac{1}{\ell^2}\left(g^{(0)}_{kj} g^{im}_{(2)} g^{(0)}_{ml} - k \leftrightarrow l\right)\right] + \cdots\,. \end{aligned} \tag{A.9}$$

[7] We neglect a possible term that is linear in η; it does not play any role in our considerations.

B. Explicit variations of the Chern-Simons Term

In this appendix we give some details on the computation of the contribution to the boundary stress tensor from the Chern-Simons term (5.1). The variation with respect to the metric is

$$\delta S_{CS}(\Gamma) = 2\int_{\mathcal{M}} \mathrm{Tr}(\delta\Gamma \wedge R) - \int_{\partial\mathcal{M}} \mathrm{Tr}(\Gamma \wedge \delta\Gamma)\,. \tag{B.1}$$

In components, the first term reads

$$\begin{aligned} \mathrm{Tr}(\delta\Gamma \wedge R) &= \delta\Gamma^{\alpha}{}_{\beta} \wedge R^{\beta}{}_{\alpha} \\ &= \delta\Gamma^{\alpha}{}_{\beta\gamma}\,\frac{1}{2}R^{\beta}{}_{\alpha\mu\nu}dx^{\gamma} \wedge dx^{\mu} \wedge dx^{\mu} \\ &= \frac{1}{2}g^{\alpha\rho}\left(\nabla_{\gamma}\delta g_{\beta\rho} + \nabla_{\beta}\delta g_{\gamma\rho} - \nabla_{\rho}\delta g_{\beta\gamma}\right)\frac{1}{2}R^{\beta}{}_{\alpha\mu\nu}\epsilon^{\gamma\mu\nu}\sqrt{g}d^{3}x \\ &= \frac{1}{2}(\nabla_{\beta}\delta g_{\gamma\rho})R^{\beta\rho}{}_{\mu\nu}\epsilon^{\gamma\mu\nu}\sqrt{g}d^{3}x\,. \end{aligned} \tag{B.2}$$

After integration by parts we thus find a bulk term of the form

$$\delta S_{CS}(\Gamma) = -\int_{\mathcal{M}} \delta g_{\gamma\rho}(\nabla_{\beta}R^{\beta\rho}{}_{\mu\nu})\epsilon^{\gamma\mu\nu} + \cdots\,, \tag{B.3}$$

which contributes to the bulk Einstein equation. This term vanishes in situations where the curvature is covariantly constant, such as in AdS_3 or BTZ. Since we're only interested in the stress tensor, the bulk term (B.3) will play no further role. The boundary term arising from the partial integration takes the form

$$\delta S_{CS}(\Gamma) = \int_{\partial\mathcal{M}} d^{2}x\sqrt{g}\;\delta g_{ij}R^{\eta j}{}_{\mu\nu}\epsilon^{i\mu\nu} = 2\int_{\partial\mathcal{M}} d^{2}x\sqrt{g}\;\delta g_{ij}R^{\eta j}{}_{\eta k}\epsilon^{i\eta k} = 0\,. \tag{B.4}$$

This term vanishes identically because, according to (A.9), we have $R^{\eta j}{}_{\eta k} \propto \delta^{j}_{k}$ up to terms that vanish as the boundary is taken to infinity.

The second term in (B.1) is explicitly a boundary term. We begin by expanding in intrinsic and extrinsic quantities as

$$\begin{aligned} \delta S_{CS}(\Gamma) &= -\int_{\partial\mathcal{M}} \Gamma^{\alpha}_{\beta i}\delta\Gamma^{\beta}_{\alpha j}dx^{i} \wedge dx^{j} \\ &= -\int_{\partial\mathcal{M}} \left[\Gamma^{\eta}_{ki}\delta\Gamma^{k}_{\eta j} + \Gamma^{k}_{\eta i}\delta\Gamma^{\eta}_{kj} + \Gamma^{k}_{li}\delta\Gamma^{l}_{kj}\right]\epsilon^{ij}\sqrt{g}d^{2}x \\ &= \int_{\partial\mathcal{M}} \left[2K^{k}_{i}\delta K_{kj} - \Gamma^{k}_{li}\delta\Gamma^{l}_{kj}\right]\epsilon^{ij}\sqrt{g}d^{2}x\,. \end{aligned} \tag{B.5}$$

Using (A.8) we find the Fefferman-Graham expansion of the first term

$$K^{i}_{k}\delta K_{ij}\epsilon^{kj} = -\frac{1}{\ell^{2}}g^{il}_{(2)}g^{(0)}_{lk}\delta g^{(0)}_{ij}\epsilon^{kj} + \cdots\,, \tag{B.6}$$

which amounts to the contribution

$$t^{ij} = \frac{2}{\ell^{2}}\left[g^{il}_{(2)}g^{(0)}_{lk}\epsilon^{jk} + i \leftrightarrow j\right]\,, \tag{B.7}$$

to the stress tensor. This contribution depends on $g_{(2)}$ and so on the state. The second term in (B.5) is finite for large η and, in the limit, depends only on $g_{(0)}$. Moreover, the variation considered in this appendix was general and we have seen explicitly that no $\delta g_{(2)}$ dependence remains for large η.

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